

POLYNOMIAL CONSERVED QUANTITIES FOR CONSTRAINED WILLMORE SURFACES

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ABSTRACT. We define a hierarchy of special classes of constrained Willmore surfaces by means of the existence of a polynomial conserved quantity of some type, filtered by an integer. Type 1 with parallel top term characterises parallel mean curvature surfaces and, in codimension 1, type 1 characterises constant mean curvature surfaces. We show that this hierarchy is preserved under both spectral deformation and Bäcklund transformation, for special choices of parameters, defining, in particular, transformations of constant mean curvature surfaces into new ones, with preservation of the mean curvature, in the latter case.

1. INTRODUCTION

Willmore surfaces are the extremals of the Willmore functional. A larger class arises when one imposes the weaker requirement that a surface extremize the Willmore functional only with respect to infinitesimally conformal variations: these are the constrained Willmore surfaces. Constrained Willmore surfaces in the conformal sphere are characterised [1, 2, 5, 9, 13, 19] by the (possibly perturbed) harmonicity of the central sphere congruence (following the terminology introduced in [10]). The theory of harmonic maps then applies and, in particular, the machinery of integrable systems becomes available. First of all, a zero-curvature representation is established: a constrained Willmore surface comes [5] equipped with an associated family d^λ of flat connections, depending on a spectral parameter $\lambda \in \mathbb{C} \setminus \{0\}$. This structure gives rise to two kinds of symmetries: a spectral deformation [5, 9, 10], by exploiting a scaling freedom in the spectral parameter, and Bäcklund transformations [10], which arise by applying chosen gauge transformations to the family of flat connections.

Alike what happens in the case of constrained Willmore surfaces, the isothermic surface condition amounts [7] just as well to the flatness of a certain family ∇^t of connections, indexed in \mathbb{R} . In [11], the classical notion of special isothermic surface, introduced by Darboux in connection with deformations of quadrics, is given a simple explanation in terms of the integrable systems approach to isothermic surfaces. They are realised as a particular case of a hierarchy of classes of isothermic surfaces filtered by an integer d . Here is the basic idea: The theory of ordinary differential equations ensures that we can find ∇^t -parallel sections depending smoothly on the spectral parameter t . The existence of such sections with polynomial dependence of degree d on t is of particular geometric significance, as first observed by F. E. Burstall and D. Calderbank (see the forthcoming paper "Conformal submanifold geometry IV-V"), and gave rise to the notion of polynomial conserved quantity of type d , developed in [11], in the isothermic context, where the notion of special isothermic surface of type d is introduced, having the classical notion as a particular case ($d = 2$).

We are in this way led to the central idea of this article, that of *special constrained Willmore surface of type d* , a constrained Willmore surface admitting some *polynomial conserved quantity of type d* in the constrained Willmore context, that is, a

certain family $p(\lambda)$ of d^λ -parallel sections with Laurent polynomial dependence on λ , with degree smaller or equal to d .

At the intersection of the class of constrained Willmore surfaces with the class of isothermic surfaces lies, in particular, that of parallel mean curvature vector surfaces in space-forms and, in particular, that of constant mean curvature surfaces in 3-dimensional space-forms. In the isothermic context, type 1 characterises [6, 11] the H -generalised surfaces in space-forms. We prove that, in the constrained Willmore context, type 1 with parallel top term characterises parallel mean curvature vector surfaces in space-forms. It follows, in particular, that, in codimension 1, type 1 characterises constant mean curvature surfaces, in both contexts.

We prove that the class of constrained Willmore surfaces of any given type is preserved by both spectral deformation and Bäcklund transformation, for special choices of parameters. Both constrained Willmore spectral deformation and Bäcklund transformation prove to preserve also the parallelism of the top term of a polynomial conserved quantity. For the particular case of type 1, this defines transformations of parallel mean curvature surfaces into new ones, and, in the particular case of codimension 1, transformations of constant mean curvature surfaces into new ones, with preservation of both the space-form and the mean curvature, under Bäcklund transformation.

Our theory is local and, throughout the text, with no need for further reference, restriction to a suitable non-empty open set shall be underlying.

The results of this paper are based, in part, on those in the first author's PhD thesis [16, 17] and some of them were announced in [18].

Acknowledgements. The authors would like to thank Rui Pacheco for helpful conversations. Very special thanks are due to Fran Burstall and David Calderbank, who first observed the particular geometric significance of polynomial dependence on a parameter for some families of parallel sections and who have had a decisive influence on the origin of this paper.

2. CONSTRAINED WILLMORE SURFACES IN THE CONFORMAL n -SPHERE

Consider $\mathbb{C}^{n+2} = \Sigma \times (\mathbb{R}^{n+1,1})^\mathbb{C}$ provided with the complex bilinear extension of the metric on $\mathbb{R}^{n+1,1}$. In what follows, we may abuse notation and make no explicit distinction between a bundle and its complexification. Throughout this text, we consider the identification

$$\wedge^2 \mathbb{R}^{n+1,1} \cong o(\mathbb{R}^{n+1,1})$$

of the exterior power $\wedge^2 \mathbb{R}^{n+1,1}$ with the orthogonal algebra $o(\mathbb{R}^{n+1,1})$ via

$$u \wedge v(w) := (u, w)v - (v, w)u$$

for $u, v, w \in \mathbb{R}^{n+1,1}$.

2.1. Conformal submanifold geometry. Our study is one of surfaces in n -dimensional space-forms, with $n \geq 3$, from a conformally-invariant viewpoint. For this, we find a convenient setting in Darboux's light-cone model of the conformal n -sphere [12]. We follow the modern account presented in [4]. So contemplate the light-cone \mathcal{L} in the Lorentzian vector space $\mathbb{R}^{n+1,1}$ and its projectivisation $\mathbb{P}(\mathcal{L})$, provided with the conformal structure defined by a metric g_σ arising from a never-zero section σ of the tautological bundle $\pi : \mathcal{L} \rightarrow \mathbb{P}(\mathcal{L})$ via $g_\sigma(X, Y) = (d\sigma(X), d\sigma(Y))$. For $v_\infty \in \mathbb{R}_\infty^{n+1,1}$, set

$$S_{v_\infty} := \{v \in \mathcal{L} : (v, v_\infty) = -1\},$$

an n -dimensional submanifold $\mathbb{R}^{n+1,1}$ which inherits from $\mathbb{R}^{n+1,1}$ a positive definite metric of (constant) sectional curvature $-(v_\infty, v_\infty)$. By construction, the bundle projection π restricts to give a conformal diffeomorphism $\pi|_{S_{v_\infty}} : S_{v_\infty} \rightarrow \mathbb{P}(\mathcal{L}) \setminus \mathbb{P}(\mathcal{L} \cap \langle v_\infty \rangle^\perp)$. In particular, choosing v_∞ to be time-like identifies $\mathbb{P}(\mathcal{L})$ with the conformal n -sphere,

$$S^n \cong \mathbb{P}(\mathcal{L}).$$

For us, a mapping $\Lambda : \Sigma \rightarrow \mathbb{P}(\mathcal{L})$, of a surface Σ , is the same as a null line subbundle of the trivial bundle $\underline{\mathbb{R}}^{n+1,1} = \Sigma \times \mathbb{R}^{n+1,1}$. Let then $\Lambda : \Sigma \rightarrow \mathbb{P}(\mathcal{L})$ be an immersion of an oriented surface Σ , which we provide with the conformal structure \mathcal{C}_Λ induced by Λ and with the canonical complex structure. Set

$$\Lambda^{1,0} := \Lambda \oplus d\sigma(T^{1,0}M), \quad \Lambda^{0,1} := \Lambda \oplus d\sigma(T^{0,1}M),$$

defined independently of the choice of $\sigma \in \Gamma(\Lambda)$ never-zero, and then $\Lambda^{(1)} := \Lambda^{1,0} + \Lambda^{0,1}$. Let $S : \Sigma \rightarrow \mathcal{G} := \text{Gr}_{(3,1)}(\mathbb{R}^{n+1,1})$ be the central sphere congruence of Λ ,

$$S = \Lambda^{(1)} \oplus \langle \Delta\sigma \rangle,$$

for σ a lift of Λ and $\Delta\sigma$ the Laplacian of σ , with respect to the metric g_σ . We have a decomposition $\underline{\mathbb{R}}^{n+1,1} = S \oplus S^\perp$ and then a decomposition of the trivial flat connection d on $\underline{\mathbb{R}}^{n+1,1}$ as

$$d = \mathcal{D} \oplus \mathcal{N},$$

for \mathcal{D} the connection given by the sum of the connections induced on S and S^\perp by d . At times, it will be convenient to make an explicit reference to the surface Λ , writing S_Λ , \mathcal{D}_Λ (or, equivalently, \mathcal{D}_S) and \mathcal{N}_Λ (or, equivalently, \mathcal{N}_S) for S , \mathcal{D} and \mathcal{N} , respectively. For later reference, we define analogously $\mathcal{D}^{\hat{d}}$ and $\mathcal{N}^{\hat{d}}$, for a general connection \hat{d} on $\underline{\mathbb{C}}^{n+2}$.

2.2. Constrained Willmore surfaces and flat connections. Willmore surfaces are characterised [1, 13, 19] by the harmonicity of the central sphere congruence. More generally:

Theorem 2.1. [2, 5, 9] *Λ is a constrained Willmore surface if and only if there exists a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ with*

$$(2.1) \quad d^{\mathcal{D}}q = 0$$

such that

$$(2.2) \quad d^{\mathcal{D}} * \mathcal{N} = 2[q \wedge * \mathcal{N}],$$

where $[\cdot, \cdot]$ denotes the 2-form defined from the Lie Bracket $[\cdot, \cdot]$ in $\mathfrak{o}(\mathbb{R}^{n+1,1})$.

The introduction of a constraint in the variational problem equips surfaces Λ with *Lagrange multipliers* q , defining pairs (Λ, q) . Willmore surfaces are the constrained Willmore surfaces admitting the zero multiplier.

For maps into a Grassmannian, harmonicity amounts [22] to the flatness of a family of connections. Ultimately, and crucially, a zero-curvature characterisation of constrained Willmore surfaces follows:

Theorem 2.2. [5] *Given a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$, (Λ, q) is a constrained Willmore surface if and only if the connection*

$$(2.3) \quad d_q^\lambda := \mathcal{D} + \lambda \mathcal{N}^{1,0} + \lambda^{-1} \mathcal{N}^{0,1} + (\lambda^2 - 1)q^{1,0} + (\lambda^{-2} - 1)q^{0,1}$$

is flat, for all $\lambda \in \mathbb{C} \setminus \{0\}$.

At times, it will be convenient to make an explicit reference to the surface Λ , writing $d_\Lambda^{\lambda,q}$ (or, equivalently, $d_S^{\lambda,q}$) for d_q^λ .

The isothermic surface condition amounts just as well to the flatness of a certain family of connections:

Proposition 2.3. [7, 15] *Λ is isothermic if and only if there exists a non-zero 1-form $\eta \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ such that $d\eta = 0$ or, equivalently, such that the connection*

$$\nabla^t := d + t\eta$$

is flat, for all $t \in \mathbb{R}$.

In the conditions of Proposition 2.3, we will say that (Λ, η) is an isothermic surface. In the case Λ is not contained in any 2-sphere, the form η is unique up to a (non-zero) constant real scale, cf. [4]. As for a given constrained Willmore surface, the multiplier q is, in general, unique, the exception being when Λ is, in addition, isothermic:

Proposition 2.4. [10] *A constrained Willmore surface admits a unique multiplier if and only if it is not an isothermic surface. Furthermore: if (Λ, η) is an isothermic constrained Willmore surface admitting a multiplier q , then the set of multipliers to Λ is the affine space $q + \langle *\eta \rangle_{\mathbb{R}}$.*

2.3. Transformations of constrained Willmore surfaces. Constrained Willmore surfaces come equipped with a family of flat connections. Transformations of this family have been exploited [5, 9, 10] in order to produce new constrained Willmore surfaces, as we recall next.

Suppose (Λ, q) be a constrained Willmore surface.

2.3.1. Spectral deformation. For each λ in S^1 , d_q^λ is a real flat metric connection, which establishes the existence of an isometry $\phi_q^\lambda : (\mathbb{R}^{n+1,1}, d_q^\lambda) \rightarrow (\mathbb{R}^{n+1,1}, d)$ of bundles, defined on a simply connected component of Σ , preserving connections, unique up to a Möbius transformation. We define a spectral deformation of Λ into new constrained Willmore surfaces by setting, for each λ in S^1 ,

$$\Lambda_q^\lambda := \phi_q^\lambda \Lambda,$$

cf. [5, 9, 10].

2.3.2. Bäcklund transformation. In [10], a version of the Terng-Uhlenbeck dressing action [20] is used to construct new constrained Willmore surfaces from a given one, as follows. Let $\rho \in \Gamma(\mathbb{R}^{n+1,1})$ be reflection across S , $\rho = \pi_S - \pi_{S^\perp}$, for π_S and π_{S^\perp} the orthogonal projections of $\mathbb{R}^{n+1,1}$ onto S and S^\perp , respectively. Let $\alpha \in \mathbb{C} \setminus S^1$ be non-zero and $L \subset \mathbb{C}^{n+2}$ be a null line bundle such that L and ρL are never orthogonal. Define transformations $p_{\alpha,L}^{(-)}(\lambda) \in \Gamma(O(\mathbb{C}^{n+2}))$, for $\lambda \in \mathbb{C} \setminus \{\pm\alpha\}$, by¹

$$p_{\alpha,L}^{(-)}(\lambda) = \begin{cases} (-)^{\frac{\lambda-\alpha}{\lambda+\alpha}} & \text{on } L; \\ 1 & \text{on } (L \oplus \rho L)^\perp; \\ (-)^{\frac{\lambda+\alpha}{\lambda-\alpha}} & \text{on } \rho L; \end{cases}$$

respectively. We define in this way two maps of $\mathbb{C} \setminus \{\pm\alpha\}$ into $\Gamma(O(\mathbb{C}^{n+2}))$ that extend holomorphically to the Riemann sphere except $\pm\alpha$, by setting $p_{\alpha,L}(\infty) := I$ and

$$p_{\alpha,L}^-(\infty) = \begin{cases} -1 & \text{on } L; \\ 1 & \text{on } (L \oplus \rho L)^\perp; \\ -1 & \text{on } \rho L. \end{cases}$$

¹In [10], an extra factor is introduced in the eigenvalues of $p_{\alpha,L}(\lambda)$, resulting in the normalization of the family $\lambda \mapsto p_{\alpha,L}(\lambda)$, $p_{\alpha,L}(1) = I$.

It will be useful to note that $p_{\alpha,L}(\lambda) = p_{\alpha^{-1},L}(\lambda^{-1})$, for all $\lambda \in \mathbb{C} \setminus \{0, \pm\alpha\}$. For further reference, observe also that

$$(2.4) \quad p_{\alpha,L}^{(-)}(-\lambda) = \rho p_{\alpha,L}^{(-)}(\lambda) \rho^{-1},$$

respectively, for all $\lambda \in \mathbb{P}^1 \setminus \{\pm\alpha\}$. Set $\hat{\alpha} := \bar{\alpha}^{-1}$, $\tilde{L} = p_{\hat{\alpha},\tilde{L}}(\alpha)L$ and $r = p_{\alpha,\tilde{L}}^{-1} p_{\hat{\alpha},\tilde{L}}$. Consider the transform

$$\hat{S} := r(1)^{-1}S$$

of S and the transforms

$$\hat{\Lambda}^{1,0} := r(1)^{-1}r(0)\Lambda^{1,0}, \quad \hat{\Lambda}^{0,1} := r(1)^{-1}r(\infty)\Lambda^{0,1},$$

of $\Lambda^{1,0}$ and $\Lambda^{0,1}$, respectively, by r . Set $\hat{\Lambda} := \hat{\Lambda}^{1,0} \cap \hat{\Lambda}^{0,1}$.

Theorem 2.5. [10] $\hat{\Lambda}$ is a surface with central sphere congruence \hat{S} . Define, furthermore, $\tilde{q} \in \Omega^1(\wedge^2 S \oplus \wedge^2 S^\perp)$ by

$$\tilde{q}^{1,0} := r(\infty)q^{1,0}r(\infty)^{-1}, \quad \tilde{q}^{0,1} := r(0)q^{0,1}r(0)^{-1}$$

and set

$$\hat{q} := r(1)^{-1}\tilde{q}r(1).$$

Then $(\hat{\Lambda}, \hat{q})$ is a constrained Willmore surface.

$(\hat{\Lambda}, \hat{q})$ is said to be the *Bäcklund transform* of (Λ, q) of parameters α, L .²

3. SPECIAL CONSTRAINED WILLMORE SURFACES OF TYPE d

In [11], the concept of special isothermic surface of type d is introduced, as an isothermic surface admitting a polynomial conserved quantity of degree d , having the classical notion of special isothermic surface as the particular case of $d = 2$. The focus of this article will be the study of *special constrained Willmore surfaces* and, in particular, that of a polynomial conserved quantity in the constrained Willmore context.

Let Λ be a surface in $S^n \cong \mathbb{P}(\mathcal{L})$.

3.1. Polynomial conserved quantities for constrained Willmore surfaces. Suppose (Λ, q) be a constrained Willmore surface, with its associated family of flat connections d_q^λ , indexed by $\mathbb{C} \setminus \{0\}$.

Definition 3.1. Let $p(\lambda) = \sum_{k=-d}^d p_k \lambda^k$ be a Laurent polynomial with coefficients in $\Gamma(\underline{\mathbb{C}}^{n+2})$, with $d \in \mathbb{N}_0$, such that:

$$(3.1a) \quad p_{-k} = \overline{p_k}, \text{ for all } k;$$

$$(3.1b) \quad p_d \in \Gamma(S^\perp);$$

$$(3.1c) \quad p_k \in \Gamma(S^\perp) \text{ if } k \text{ and } d \text{ have the same parity; otherwise } p_k \in \Gamma(S);$$

$$(3.1d) \quad p(1) \neq 0.$$

We say that $p(\lambda)$ is a *polynomial conserved quantity* of (Λ, q) of type d if

$$(3.2) \quad d_q^\lambda p(\lambda) = 0,$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$. We say that (Λ, q) is a *special constrained Willmore surface of type d* if it admits a polynomial conserved quantity of type d .

Constrained Willmore surfaces in space-forms constitute a conformally-invariant class of surfaces and so does the class of special constrained Willmore surfaces of type d , given $d \in \mathbb{N}_0$:

²We are, in fact, considering the reparametrization of the Bäcklund transformation presented in [10] that results of interchanging parameters α, L with parameters $\hat{\alpha}, \tilde{L}$.

Proposition 3.2. *Let d be in \mathbb{N}_0 and T be in $O(\mathbb{R}^{n+1,1})$. Suppose (Λ, q) is a constrained Willmore surface of type d . Then so is $(T\Lambda, Ad_T(q))$.*

Proof. The fact that $\mathcal{D}_{T\Lambda} = T \circ \mathcal{D}_\Lambda \circ T^{-1}$ and $\mathcal{N}_{T\Lambda} = T \circ \mathcal{N}_\Lambda \circ T^{-1}$ makes it clear that $(T\Lambda, Ad_T(q))$ is a constrained Willmore surface and, furthermore, that, if $p(\lambda) = \sum_{k=-d}^d p_k \lambda^k$ is a polynomial conserved quantity of (Λ, q) of type d , then

$$s(\lambda) := T \circ p(\lambda) = \sum_{k=-d}^d T(p_k) \lambda^k$$

is a polynomial conserved quantity of $(T\Lambda, Ad_T(q))$ of type d . \square

Proposition 3.3. *A Laurent polynomial $p(\lambda) = \sum_{k=-d}^d p_k \lambda^k$ satisfying the conditions (3.1) is a polynomial conserved quantity of (Λ, q) if and only if*

$$(3.3) \quad \mathcal{D}p_k + \mathcal{N}^{1,0}p_{k-1} + \mathcal{N}^{0,1}p_{k+1} + q^{1,0}p_{k-2} + q^{0,1}p_{k+2} - qp_k = 0, \quad \forall k \in \{0, \dots, d+2\},$$

with the convention

$$p_{-d-4} = p_{-d-3} = p_{-d-2} = p_{-d-1} = p_{d+1} = p_{d+2} = p_{d+3} = p_{d+4} = 0.$$

Proof. The result is a direct consequence of the fact that $d^{\lambda, q}p(\lambda) = 0$ if and only if

$$\begin{aligned} & \sum_{k=-d}^d \mathcal{D}p_k \lambda^k + \sum_{k=-d+1}^{d+1} \mathcal{N}^{1,0}p_{k-1} \lambda^k + \sum_{k=-d-1}^{d-1} \mathcal{N}^{0,1}p_{k+1} \lambda^k + \\ & \sum_{k=-d+2}^{d+2} q^{1,0}p_{k-2} \lambda^k + \sum_{k=-d-2}^{d-2} q^{0,1}p_{k+2} \lambda^k + \sum_{k=-d}^d (-q^{1,0} - q^{0,1})p_k \lambda^k = 0, \end{aligned}$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$. \square

Proposition 3.4. *If $p(\lambda) = \sum_{k=-d}^d p_k \lambda^k$ is a polynomial conserved quantity of (Λ, q) of type $d \in \mathbb{N}_0$, then*

- (1) p_0 is real;
- (2) $p(1)$ is real and constant;
- (3) $\mathcal{D}^{0,1}p_d = 0$;
- (4) $\mathcal{D}^{1,0}p_d + \mathcal{N}^{1,0}p_{d-1} = 0$;
- (5) $\mathcal{N}^{1,0}p_d + q^{1,0}p_{d-1} = 0$;
- (6) the Laurent polynomial $(p(\lambda), p(\lambda))$ has constant (complex) coefficients.

Proof. The reality of p_0 and $p(1)$ is a consequence of $\overline{p_0} = p_0$ together with $p(1) = p_0 + 2 \sum_{k=1}^d \operatorname{Re}(p_k)$. The constancy of $p(1)$ is immediate from evaluating $d_q^\lambda p(\lambda)$ at $\lambda = 1$. On the other hand, for $k = d$, equation (3.3) establishes $\mathcal{D}^{0,1}p_d = 0$ and $\mathcal{D}^{1,0}p_d + \mathcal{N}^{1,0}p_{d-1} = 0$ (note that $qp_d = 0$ and $qp_{d-2} = 0$, as $p_d, p_{d-2} \in \Gamma(S^\perp)$), whereas, taking $k = d+1$, we get $\mathcal{N}^{1,0}p_d + q^{1,0}p_{d-1} = 0$. The fact that d_q^λ is a metric connection, together with equation (3.2), establishes

$$d(p(\lambda), p(\lambda)) = 2(d_q^\lambda p(\lambda), p(\lambda)) = 0,$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$, showing that the polynomial $(p(\lambda), p(\lambda))$ has constant coefficients and completing the proof. \square

According to Proposition 3.4, the existence of a polynomial conserved quantity $p(\lambda)$ establishes, in particular, $p(1)$ as a non-zero real vector in $\mathbb{R}^{n+1,1}$, defining therefore a space-form $S_{p(1)}$, which will be of particular geometric relevance, as we shall see later.

Proposition 3.5. *If (Λ, q) admits a polynomial conserved quantity $p(\lambda)$ of type d , then it admits a polynomial conserved quantity $s(\lambda)$ of type $d+1$ with $s(1) = p(1)$.*

Proof. Take

$$s(\lambda) := \frac{1}{2}(\lambda^{-1} + \lambda)p(\lambda) = \frac{1}{2} \sum_{k=-d-1}^{d+1} (p_{k-1} + p_{k+1})\lambda^k,$$

for $p(\lambda) = \sum_{k=-d}^d p_k \lambda^k$ and under the convention $p_{-d-2} = p_{-d-1} = p_{d+1} = p_{d+2} = 0$. \square

3.2. Non-full constrained Willmore surfaces. We say that a surface in S^n is *full* if it does not lie in any proper sub-sphere of S^n . In the isothermic context, type 0 characterises [11] the surfaces which are not full. This is also the case in the constrained Willmore context:

Proposition 3.6. *(Λ, q) is a special constrained Willmore surface of type 0 if and only if Λ is not full.*

Proof. Suppose (Λ, q) admits a polynomial conserved quantity $p(\lambda) = p_0$. Then p_0 is a non-zero, real constant section of S^\perp and Λ lies therefore in the $(n-1)$ -sphere $\mathbb{P}(\mathcal{L} \cap \langle p_0 \rangle^\perp)$. Conversely, if Λ takes values in some sub-sphere, say $\mathbb{P}(\mathcal{L} \cap \langle u \rangle^\perp)$, where u is a positive definite vector of $\mathbb{R}^{n+1,1}$, we have $u \in \Gamma(S^\perp)$. Hence $p(\lambda) := u$ satisfies the conditions (3.1) and, for all $\lambda \in \mathbb{C} \setminus \{0\}$,

$$d_q^\lambda p(\lambda) = (\mathcal{D} + \lambda \mathcal{N}^{1,0} + \lambda^{-1} \mathcal{N}^{0,1})p_0 = 0,$$

as $p_0 \in \Gamma(S^\perp)$ and $dp_0 = 0$ (or, equivalently, $\mathcal{D}p_0 = 0 = \mathcal{N}p_0$). \square

3.3. Surfaces with parallel mean curvature vector. In the isothermic context, type 1 characterises [6] (see also [11]), in general, the H -generalised surfaces in some space-form. In the constrained Willmore context, type 1 with parallel top term characterises surfaces with parallel mean curvature vector, as we shall see in this section. In particular, for codimension 1, type 1 characterises surfaces with constant mean curvature vector in both contexts.

Given $v_\infty \in \mathbb{R}^{n+1,1}$ such that $v_\infty \notin \Gamma(\Lambda^\perp)$ and $\sigma \in \Gamma(\Lambda)$ never-zero, Λ defines a local immersion

$$\sigma_\infty := (\pi|_{S_{v_\infty}})^{-1} \circ \Lambda = -\frac{1}{(\sigma, v_\infty)} \sigma : \Sigma \rightarrow S_{v_\infty},$$

of Σ into the space-form S_{v_∞} . Let v_∞^T and v_∞^\perp denote the orthogonal projections of v_∞ onto S and S^\perp , respectively. Consider the normal bundle $V_{v_\infty}^\perp = (\Lambda^{(1)} \oplus \langle v_\infty \rangle)^\perp$ of σ_∞ and let \mathbf{H} denote the mean curvature vector of σ_∞ . The map $\mathcal{Q} : V_{v_\infty}^\perp \rightarrow S^\perp$ defined by

$$\mathcal{Q} : \xi \mapsto (\mathbf{H}, \xi)\sigma_\infty + \xi$$

is an isomorphism of bundles preserving connections. Note that, as

$$(v_\infty^\perp, (\mathbf{H}, \xi)\sigma_\infty + \xi) = -(\mathbf{H}, \xi), \quad \forall \xi \in \Gamma(V_{v_\infty}^\perp)$$

we have

$$(3.4) \quad \mathcal{Q}\mathbf{H} = -v_\infty^\perp$$

and then $\nabla^{V_{v_\infty}^\perp} \mathbf{H} = 0$ (i.e., \mathbf{H} parallel) if and only if $\mathcal{D}v_\infty^\perp = 0$.

In this section, we first recall that parallel mean curvature vector surfaces are, indeed, examples of constrained Willmore surfaces, proving it in our setting and providing a multiplier (which is certainly not the only multiplier, as parallel mean curvature vector surfaces are examples of isothermic surfaces).

Proposition 3.7. *Suppose Λ has parallel mean curvature vector in some space-form S_{v_∞} . Then (Λ, q_∞) is a constrained Willmore surface, where*

$$q_\infty := \frac{1}{2} \sigma_\infty \wedge \mathcal{N} v_\infty^\perp = \frac{1}{2} \sigma_\infty \wedge dv_\infty^\perp.$$

Proof. First of all, note that the real 1-form q takes values in $\Lambda \wedge \Lambda^{(1)}$, as $\mathcal{N} v_\infty^\perp$ lives in S and

$$(\mathcal{N} v_\infty^\perp, \sigma_\infty) = -(v_\infty^\perp, \mathcal{N} \sigma_\infty) = -(v_\infty^\perp, 0) = 0.$$

Taking into account that

$$\begin{aligned} 2d^\mathcal{P} q_\infty \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) &= (\sigma_\infty)_z \wedge (v_\infty^\perp)_{\bar{z}} - (\sigma_\infty)_{\bar{z}} \wedge (v_\infty^\perp)_z + \sigma_\infty \wedge \pi_S((v_\infty^\perp)_{\bar{z}z} - (v_\infty^\perp)_{z\bar{z}}) \\ &= (\sigma_\infty)_z \wedge (v_\infty^\perp)_{\bar{z}} - (\sigma_\infty)_{\bar{z}} \wedge (v_\infty^\perp)_z, \end{aligned}$$

for π_S the orthogonal projection of $\mathbb{R}^{n+1,1}$ onto S , and noting that $(v_\infty^\perp)_z \in \Gamma(\langle (\sigma_\infty)_{\bar{z}} \rangle)$ and $(v_\infty^\perp)_{\bar{z}} \in \Gamma(\langle (\sigma_\infty)_z \rangle)$, by virtue of $(v_\infty^\perp)_z$ being orthogonal to v_∞ and $(\sigma_\infty)_{\bar{z}}$, and $(v_\infty^\perp)_{\bar{z}}$ being orthogonal to v_∞ and $(\sigma_\infty)_z$, we conclude that $d^\mathcal{P} q_\infty = 0$.

Now let us prove that $d^\mathcal{P} * \mathcal{N} = 2[q_\infty \wedge * \mathcal{N}]$. For that, first observe that $d^\mathcal{P} * \mathcal{N}$ and $[q_\infty \wedge * \mathcal{N}]$ are determined by the respective restrictions to S , as both are 2-forms taking values in $S \wedge S^\perp$. It is therefore enough to prove the equality in S .

For each $\xi \in \Gamma(\mathbb{R}^{n+1,1})$,

$$d^\mathcal{P} * \mathcal{N} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \xi = 2i(\mathcal{D}_z(\mathcal{N}_{\bar{z}} \xi) - \mathcal{N}_{\bar{z}}(\mathcal{D}_z \xi) + \mathcal{D}_{\bar{z}}(\mathcal{N}_z \xi) - \mathcal{N}_z(\mathcal{D}_{\bar{z}} \xi)).$$

In view of the flatness of d , characterised by

$$R^\mathcal{P} + d^\mathcal{P} \mathcal{N} + \frac{1}{2} [\mathcal{N} \wedge \mathcal{N}] = 0,$$

and encoding, in particular, $d^\mathcal{P} \mathcal{N} = 0$, it follows that

$$(3.5) \quad d^\mathcal{P} * \mathcal{N} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \xi = 2i(\mathcal{D}_z(\mathcal{N}_{\bar{z}} \xi) - \mathcal{N}_{\bar{z}}(\mathcal{D}_z \xi)) = 2i(\mathcal{D}_{\bar{z}}(\mathcal{N}_z \xi) - \mathcal{N}_z(\mathcal{D}_{\bar{z}} \xi)).$$

On the other hand,

$$(3.6) \quad 2[q_\infty \wedge * \mathcal{N}] \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \xi = 2i((q_\infty)_z(\mathcal{N}_{\bar{z}} \xi) + (q_\infty)_{\bar{z}}(\mathcal{N}_z \xi) - \mathcal{N}_z((q_\infty)_{\bar{z}} \xi) - \mathcal{N}_{\bar{z}}((q_\infty)_z \xi)).$$

We will start by proving that both (3.5) and (3.6) vanish, whenever $\xi \in \Gamma(\Lambda^{(1)})$.

As for (3.5), we have

$$\mathcal{D}_z(\mathcal{N}_{\bar{z}} \sigma_\infty) - \mathcal{N}_{\bar{z}}(\mathcal{D}_z \sigma_\infty) = -\mathcal{N}_{\bar{z}}((\sigma_\infty)_z) = 0$$

and, as $\mathcal{D}_z(\sigma_\infty)_z$ takes values in $\Lambda^{1,0}$,

$$\mathcal{D}_z(\mathcal{N}_{\bar{z}}(\sigma_\infty)_z) - \mathcal{N}_{\bar{z}}(\mathcal{D}_z(\sigma_\infty)_z) = -\mathcal{N}_{\bar{z}}(\mathcal{D}_z(\sigma_\infty)_z) = 0.$$

Similarly we get

$$\mathcal{D}_{\bar{z}}(\mathcal{N}_z(\sigma_\infty)_{\bar{z}}) - \mathcal{N}_z(\mathcal{D}_{\bar{z}}(\sigma_\infty)_{\bar{z}}) = 0.$$

On the other hand, since $(\mathcal{N} \xi, \sigma_\infty) = -(\xi, \mathcal{N} \sigma_\infty) = 0$, for every $\xi \in \Gamma(S^\perp)$, we have $\mathcal{N} \in \Omega^2(\Lambda^{(1)} \wedge S^\perp)$ and also $*\mathcal{N} \in \Omega^2(\Lambda^{(1)} \wedge S^\perp)$. Having in consideration that

$$[T, a \wedge b] = (Ta) \wedge b + a \wedge (Tb),$$

for all $T \in o(\mathbb{R}^{n+1,1})$ and $a, b \in \mathbb{R}^{n+1,1}$, we conclude that $[q_\infty \wedge * \mathcal{N}] \in \Omega^2(\Lambda \wedge S^\perp)$ and, therefore, that (3.6) vanishes for all $\xi \in \Lambda^{(1)}$.

The proof is therefore complete if we establish the equality between (3.5) and (3.6) for $\xi = v_\infty^T$. Since

$$\mathcal{N} v_\infty^T = -\mathcal{D} v_\infty^\perp = 0$$

and

$$\mathcal{N}_z(\mathcal{N}_{\bar{z}}v_\infty^\perp) = \mathcal{N}_z((v_\infty^\perp)_{\bar{z}}) = \pi_{S^\perp}((v_\infty^\perp)_{\bar{z}z}) = \pi_{S^\perp}((v_\infty^\perp)_{z\bar{z}}) = \mathcal{N}_{\bar{z}}(\mathcal{N}_zv_\infty^\perp),$$

we get

$$\begin{aligned} 2[q_\infty \wedge * \mathcal{N}] \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) v_\infty^T &= -i(\mathcal{N}_z((\sigma_\infty, v_\infty^T)\mathcal{N}_{\bar{z}}v_\infty^\perp) + \mathcal{N}_{\bar{z}}((\sigma_\infty, v_\infty^T)\mathcal{N}_zv_\infty^\perp)) \\ &= i(\mathcal{N}_z(\mathcal{N}_{\bar{z}}v_\infty^\perp) + \mathcal{N}_{\bar{z}}(\mathcal{N}_zv_\infty^\perp)) \\ &= 2i\mathcal{N}_z(\mathcal{N}_{\bar{z}}v_\infty^\perp) \\ &= -2i\mathcal{N}_z(\mathcal{D}_{\bar{z}}v_\infty^T) \\ &= 2i(\mathcal{D}_{\bar{z}}(\mathcal{N}_zv_\infty^T) - \mathcal{N}_z(\mathcal{D}_{\bar{z}}v_\infty^T)) \\ &= d^{\mathcal{D}} * \mathcal{N} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) v_\infty^T. \end{aligned}$$

□

Proposition 3.8. *Suppose (Λ, q) is a constrained Willmore surface. A Laurent polynomial $p(\lambda)$ satisfying the conditions (3.1) for $d = 1$ is a polynomial conserved quantity of (Λ, q) if and only if*

$$dp(1) = 0, \mathcal{D}^{0,1}p_1 = 0, \mathcal{N}^{1,0}p_1 + q^{1,0}p_0 = 0.$$

Proof. By Proposition 3.4, we are left to prove that these three equations establish $p(\lambda)$ as a polynomial conserved quantity of (Λ, q) . For that, suppose that $dp(1) = 0$, $\mathcal{D}^{0,1}p_1 = 0$ and $\mathcal{N}^{1,0}p_1 + q^{1,0}p_0 = 0$ and first note that, according to Proposition 3.3, it is enough to establish

$$\mathcal{N}^{1,0}p_1 + q^{1,0}p_0 = 0, \mathcal{D}p_1 + \mathcal{N}^{1,0}p_0 = 0$$

and

$$\mathcal{D}p_0 + \mathcal{N}^{1,0}p_{-1} + \mathcal{N}^{0,1}p_1 - qp_0 = 0.$$

Considering orthogonal projections onto S and S^\perp , $dv_\infty = 0$ gives

$$(3.7) \quad \mathcal{D}(p_{-1} + p_1) + \mathcal{N}p_0 = 0$$

and

$$(3.8) \quad \mathcal{N}(p_{-1} + p_1) + \mathcal{D}p_0 = 0.$$

From (3.7), we get $\mathcal{D}^{1,0}(p_{-1} + p_1) + \mathcal{N}^{1,0}p_0 = 0$. But $\mathcal{D}^{0,1}p_1 = 0$ or, equivalently, $\mathcal{D}^{1,0}p_{-1} = 0$. Hence $\mathcal{D}^{1,0}(p_{-1} + p_1) = \mathcal{D}p_1$ and then $\mathcal{D}p_1 + \mathcal{N}^{1,0}p_0 = 0$. Finally, in view of (3.8),

$$\mathcal{D}p_0 + \mathcal{N}^{1,0}p_{-1} + \mathcal{N}^{0,1}p_1 - qp_0 = -\mathcal{N}(p_{-1} + p_1) + \mathcal{N}^{1,0}p_{-1} + \mathcal{N}^{0,1}p_1 - qp_0,$$

which implies

$$\begin{aligned} \mathcal{D}p_0 + \mathcal{N}^{1,0}p_{-1} + \mathcal{N}^{0,1}p_1 - qp_0 &= -\mathcal{N}^{0,1}p_{-1} - \mathcal{N}^{1,0}p_1 - qp_0 \\ &= -2\operatorname{Re}(\mathcal{N}^{1,0}p_1 + q^{1,0}p_0) \\ &= 0 \end{aligned}$$

and completes the proof. □

Remark 3.9. If (Λ, q) is a constrained Willmore surface and $p(\lambda)$ is a polynomial conserved quantity of type 1 of (Λ, q) , then

$$q = \sigma_\infty \wedge (\mathcal{N}^{1,0}p_1 + \mathcal{N}^{0,1}p_{-1}) = 2\sigma_\infty \wedge \operatorname{Re}(\mathcal{N}^{1,0}p_1).$$

As a matter of fact, $q = \sigma_\infty \wedge \vartheta$, for some $\vartheta \in \Omega^1(\Lambda^{(1)})$, so that $qp_0 = -\vartheta - (\vartheta, p_0)\sigma_\infty$, as

$$(\sigma_\infty, p_0) = (\sigma_\infty, v_\infty^T) = (\sigma_\infty, v_\infty) = -1.$$

Hence

$$q = -\sigma_\infty \wedge (qp_0 + (\vartheta, p_0)\sigma_\infty) = -\sigma_\infty \wedge qp_0.$$

The result is now an immediate consequence of

$$qp_0 = q^{1,0}p_0 + q^{0,1}p_0 = -\mathcal{N}^{1,0}p_1 - \mathcal{N}^{0,1}p_{-1} = -2\operatorname{Re}(\mathcal{N}^{1,0}p_1).$$

Theorem 3.10. *Λ is a constrained Willmore surface admitting a polynomial conserved quantity $p(\lambda)$ of type 1 with parallel top term and $p(1) = v_\infty$ if and only if the surface σ_∞ defined by Λ in S_{v_∞} has parallel mean curvature vector.*

Proof. Suppose first that the surface Λ admits parallel mean curvature vector in the space-form S_{v_∞} . According to Proposition 3.7, we get then the constrained Willmore surface (Λ, q_∞) , for $q_\infty := \frac{1}{2}\sigma_\infty \wedge \mathcal{N}v_\infty^\perp$. Considering

$$(3.9) \quad p(\lambda) := \frac{1}{2}v_\infty^\perp \lambda^{-1} + v_\infty^T + \frac{1}{2}v_\infty^\perp \lambda,$$

we have $p(1) = v_\infty$ constant and $\mathcal{D}p_1 = \frac{1}{2}\mathcal{D}v_\infty^\perp = 0$. Furthermore,

$$\mathcal{N}^{1,0}p_1 + (q_\infty)^{1,0}p_0 = 0,$$

as $\mathcal{N}v_\infty^T = -\mathcal{D}v_\infty^\perp = 0$, and then

$$\mathcal{N}p_1 + q_\infty p_0 = \frac{1}{2}(\mathcal{N}v_\infty^\perp + (\sigma_\infty, v_\infty^T)\mathcal{N}v_\infty^\perp - (\mathcal{N}v_\infty^\perp, v_\infty^T)\sigma_\infty) = \frac{1}{2}(v_\infty^\perp, \mathcal{N}v_\infty^T)\sigma_\infty = 0.$$

Hence $p(\lambda)$ is a polynomial conserved quantity of (Λ, q_∞) with parallel top term.

Conversely, assuming that a constrained Willmore surface (Λ, q) admits a polynomial conserved quantity $p(\lambda) := p_{-1}\lambda^{-1} + p_0 + p_1\lambda$ of type 1, with p_1 parallel and $p(1) = v_\infty$, we obtain also $p_{-1} = \overline{p_1}$ parallel, and therefore

$$\mathcal{D}v_\infty^\perp = \mathcal{D}(p_{-1} + p_1) = 0.$$

Consequently, σ_∞ has parallel mean curvature vector. \square

Remark 3.11. In the proof of Theorem 3.10, we verified, in particular, that, if Λ has parallel mean curvature vector in some space-form, then Λ admits a polynomial conserved quantity of type 1 with real top term, given by (3.9). Conversely, given a polynomial conserved quantity $p(\lambda) := p_{-1}\lambda^{-1} + p_0 + p_1\lambda$ of type 1, with p_1 real and $p(1) = v_\infty$, of a constrained Willmore surface (Λ, q) , we have $\mathcal{D}^{0,1}p_1 = 0$. Since p_1 is real, we get also $\mathcal{D}^{1,0}p_1 = 0$ and then p_1 parallel. We conclude that Λ has parallel mean curvature vector in S_{v_∞} if and only if it is a constrained Willmore surface admitting a polynomial conserved quantity of type 1 with real top term and $p(1) = v_\infty$.

Corollary 3.12. *Λ is a constrained Willmore surface admitting a polynomial conserved quantity $p(\lambda)$ of type 1 with imaginary top term and $p(1) = v_\infty$ if and only if the surface σ_∞ defined by Λ in S_{v_∞} is minimal.*

Proof. First note that $\mathbf{H} = 0$ if and only if $v_\infty^\perp = 0$, by (3.4). Considering now a polynomial conserved quantity of type 1 with imaginary top term and $p(1) = v_\infty$, we get $v_\infty^\perp = \overline{p_1} + p_1 = 0$ and then σ_∞ minimal. Conversely, if σ_∞ is minimal then, according to the proof of Theorem 3.10, (Λ, q_∞) is a constrained Willmore surface, for $q_\infty = 0$, and the result follows for $p(\lambda) = v_\infty^T$. \square

Surfaces with parallel mean curvature vector (in some space-form) are characterised by the existence of a polynomial conserved quantity of type 1 with parallel top term, cf. Theorem 3.10. These surfaces are characterised, alternatively, by the existence of some polynomial conserved quantity of type 1 with real top term, as observed in Remark 3.11. In the latest characterisation, the reality of a polynomial conserved quantity depends in general on the multiplier in question. In contrast,

the parallelism of the top term of a polynomial conserved quantity is independent of the chosen multiplier, as we establish next:

Proposition 3.13. *Suppose that Λ is full. If (Λ, q) is a constrained Willmore surface admitting a polynomial conserved quantity $p(\lambda)$ of type 1 with parallel top term and $p(1) = v_\infty$, then for any multiplier q' of Λ , (Λ, q') also admits a polynomial conserved quantity $p'(\lambda)$ of type 1 with parallel top term and $p'(1) = v_\infty$.*

Proof. Suppose that (Λ, q) is a constrained Willmore surface admitting $p(\lambda) = p_{-1}\lambda^{-1} + p_0 + p_1\lambda$ as a polynomial conserved quantity of type 1 with $p(1) = v_\infty$ and parallel top term. From Theorem 3.10, we get that v_∞^\perp is a parallel section of S^\perp . Since Λ is full, v_∞^\perp cannot be constant. Consider the non-zero 1-form

$$\eta := \sigma_\infty \wedge dv_\infty^\perp = \sigma_\infty \wedge \mathcal{N}v_\infty^\perp.$$

Using the proof of Proposition 3.7, we have $\eta \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ and

$$d\eta\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = (\sigma_\infty)_z \wedge (v_\infty^\perp)_{\bar{z}} - (\sigma_\infty)_{\bar{z}} \wedge (v_\infty^\perp)_z + \sigma_\infty \wedge ((v_\infty^\perp)_{\bar{z}z} - (v_\infty^\perp)_{zz}) = 0.$$

Given a multiplier q' of Λ , $q' = q + t * \eta$, for some $t \in \mathbb{R}$. Considering the polynomial

$$p'(\lambda) = (p_{-1} + itv_\infty^\perp)\lambda^{-1} + p_0 + (p_1 - itv_\infty^\perp)\lambda,$$

we have immediately $p'(1) = p(1)$ and $\mathcal{D}(p_1 - itv_\infty^\perp) = 0$. Furthermore,

$$\begin{aligned} \mathcal{N}^{1,0}(p_1 - itv_\infty^\perp) + (q')^{1,0}p_0 &= \mathcal{N}^{1,0}(p_1 - itv_\infty^\perp) + (q^{1,0} - it\eta^{1,0})p_0 \\ &= -\mathcal{N}^{1,0}(itv_\infty^\perp) - it\eta^{1,0}v_\infty^T = 0, \end{aligned}$$

having in consideration that

$$\eta^{1,0}v_\infty^T = -\mathcal{N}^{1,0}v_\infty^\perp - (\mathcal{N}^{1,0}v_\infty^\perp, v_\infty^T)\sigma_\infty$$

and that

$$(\mathcal{N}v_\infty^\perp, v_\infty^T) = -(v_\infty^\perp, \mathcal{N}v_\infty^T) = (v_\infty^\perp, \mathcal{D}v_\infty^\perp) = 0.$$

□

In codimension 1, Theorem 3.10 reads that the constant mean curvature surfaces in some space-form are exactly the constrained Willmore surfaces which admit a polynomial conserved quantity of type 1 with parallel top term. Actually, the extra condition of parallel top term can be omitted:

Theorem 3.14. *Let $n = 3$. Then Λ is a constrained Willmore surface admitting a polynomial conserved quantity $p(\lambda)$ of type 1 with $p(1) = v_\infty$ if and only if the surface σ_∞ defined by Λ in S_{v_∞} is a constant mean curvature surface. Furthermore, if $p(\lambda) = p_{-1}\lambda^{-1} + p_0 + p_1\lambda$ is a polynomial conserved quantity of type 1 of Λ , then the constant mean curvature H of Λ in S_{v_∞} satisfies*

$$H^2 = (v_\infty^\perp, v_\infty^\perp) = 4(\operatorname{Re}(p_1), \operatorname{Re}(p_1)).$$

Proof. Suppose that Λ is a constrained Willmore surface admitting a polynomial conserved quantity $p(\lambda) = p_{-1}\lambda^{-1} + p_0 + p_1\lambda$ of type 1. Considering a unit parallel section N of S^\perp , we have in particular $S^\perp = \langle N \rangle$ (recall that codimension is 1). Take $\beta \in \Gamma(\mathbb{C})$ such that $p_1 = \beta N$. Since $\beta^2 = (p_1, p_1)$ is constant, we conclude that β is constant, which implies that $p_1 = \beta N$ is parallel.

Finally, taking into account that $v_\infty^\perp = -HN$, with $N = H\sigma_\infty + \xi$, $\xi \in \Gamma(V_{v_\infty}^\perp)$ unitary and H the constant mean curvature of σ_∞ with respect to ξ , we get automatically $H^2 = (v_\infty^\perp, v_\infty^\perp)$. □

4. TRANSFORMATIONS OF SPECIAL CONSTRAINED WILLMORE SURFACES

The class of constrained Willmore surfaces of any given type is preserved by both spectral deformation and Bäcklund transformation, for special choices of parameters, as we shall see next. Both constrained Willmore spectral deformation and Bäcklund transformation prove to preserve also the parallelism of the top term of a polynomial conserved quantity. For the particular case of type 1, this defines transformations of surfaces with parallel mean curvature into new ones (and, in the particular case of codimension 1, transformations of surfaces with constant mean curvature vector into new ones).

Let (Λ, q) be a special constrained Willmore surface of type d .

Theorem 4.1. *Let μ be in S^1 and $\phi_q^\mu : (\mathbb{R}^{n+1,1}, d_q^\mu) \rightarrow (\mathbb{R}^{n+1,1}, d)$ be an isometry of bundles, preserving connections. Suppose that $p(\lambda)$ is a polynomial conserved quantity of type d of (Λ, q) with $p(\mu)$ non-zero. Then $\phi_q^\mu p(\mu\lambda)$ is a polynomial conserved quantity of type d of the spectral deformation $(\phi_q^\mu \Lambda, \text{Ad}_{\phi_q^\mu}(q_\mu))$, of parameter μ , of Λ .*

Proof. Write $p(\lambda) = \sum_{k=-d}^d p_k \lambda^k$. By hypothesis,

$$v_\infty^\mu := \phi_q^\mu \left(\sum_{k=-d}^d p_k \mu^k \right) = \phi_q^\mu p(\mu)$$

is non-zero. On the other hand, as ϕ_q^μ is real and μ is unit, we have

$$\mu^{-k} \phi_q^\mu p_{-k} = \overline{\mu^k \phi_q^\mu p_k}.$$

Having in consideration that ϕ_q^μ is an isometry, and, in particular, $(\phi_q^\mu S)^\perp = \phi_q^\mu S^\perp$, and that

$$S_{\phi_q^\mu \Lambda} = \phi_q^\mu S_\Lambda,$$

we conclude that

$$\phi_q^\mu p(\mu\lambda) = \sum_{k=-d}^d (\mu^k \phi_q^\mu p_k) \lambda^k$$

is of the right form. The fact that $\phi_q^\mu : (\mathbb{R}^{n+1,1}, d_q^\mu) \rightarrow (\mathbb{R}^{n+1,1}, d)$ preserves connections, and, consequently,

$$d_{\phi_q^\mu \Lambda}^{\lambda, \text{Ad}_{\phi_q^\mu}(q_\mu)} = \phi_q^\mu \circ (d_\Lambda^{\mu, q})_\Lambda^{\lambda, q_\mu} \circ (\phi_q^\mu)^{-1} = \phi_q^\mu \circ d_\Lambda^{\mu\lambda, q} \circ (\phi_q^\mu)^{-1},$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$, completes the proof. \square

Bäcklund transformations of constrained Willmore surfaces preserve the existence of a polynomial conserved quantity of the same type, in the following terms:

Theorem 4.2. *Suppose $p(\lambda)$ is a polynomial conserved quantity of type d of (Λ, q) . Suppose α, L are Bäcklund transformation parameters to (Λ, q) with*

$$(4.1) \quad p(\alpha) \perp \bar{L}.$$

Then

$$\hat{p}(\lambda) := r(1)^{-1} r(\bar{\lambda}^{-1}) p(\lambda)$$

is a polynomial conserved quantity of type d of the Bäcklund transform $(\hat{\Lambda}, \hat{q})$ of (Λ, q) of parameters α, L .

To prove the theorem, we start by establishing an alternative expression for the dressing gauge r :

Lemma 4.3. *Suppose α, L are Bäcklund transformation parameters to (Λ, q) . Then*

$$(4.2) \quad r = K p_{\hat{\alpha}, \bar{L}} p_{\alpha, L}^{-1},$$

for

$$K := p_{\hat{\alpha}, \bar{L}}(0) p_{\alpha, \bar{L}}^{-1}(0).$$

The proof of the lemma we present next will be based on the following:

Lemma 4.4. [4] *Let*

$$\gamma(\lambda) = \lambda \pi_{L_1} + \pi_{L_0} + \lambda^{-1} \pi_{L_{-1}}$$

and

$$\hat{\gamma}(\lambda) = \lambda \pi_{\hat{L}_1} + \pi_{\hat{L}_0} + \lambda^{-1} \pi_{\hat{L}_{-1}}$$

be homomorphisms of \mathbb{C}^{n+2} corresponding to decompositions

$$\mathbb{C}^{n+2} = L_1 \oplus L_0 \oplus L_{-1} = \hat{L}_1 \oplus \hat{L}_0 \oplus \hat{L}_{-1}$$

with $L_{\pm 1}$ and $\hat{L}_{\pm 1}$ null lines and $L_0 = (L_1 \oplus L_{-1})^\perp$, $\hat{L}_0 = (\hat{L}_1 \oplus \hat{L}_{-1})^\perp$. Suppose $\text{Ad } \gamma$ and $\text{Ad } \hat{\gamma}$ have simple poles. Suppose as well that ξ is a map into $O(\mathbb{C}^{n+2})$ holomorphic near 0 such that $L_1 = \xi(0)\hat{L}_1$. Then $\gamma\xi\hat{\gamma}^{-1}$ is holomorphic and invertible at 0.

Next we prove Lemma 4.3:

Proof. In view of $L = p_{\hat{\alpha}, \bar{L}}(\alpha)^{-1} \tilde{L}$, after an appropriate change of variable, we conclude, by Lemma 4.4, that $p_{\alpha, L}^{-1} p_{\hat{\alpha}, \bar{L}}^{-1} (p_{\alpha, \bar{L}}^{-1})^{-1}$ admits a holomorphic and invertible extension to $\mathbb{P}^1 \setminus \{\pm \hat{\alpha}, -\alpha\}$. On the other hand, in view of (2.4), the holomorphicity and invertibility of $p_{\alpha, L}^{-1} p_{\hat{\alpha}, \bar{L}}^{-1} (p_{\alpha, \bar{L}}^{-1})^{-1}$ at the points α and $-\alpha$ are equivalent. Thus $p_{\alpha, L}^{-1} p_{\hat{\alpha}, \bar{L}}^{-1} (p_{\alpha, \bar{L}}^{-1})^{-1}$ admits a holomorphic and invertible extension to $\mathbb{P}^1 \setminus \{\pm \hat{\alpha}\}$, and so does, therefore, $(p_{\alpha, L}^{-1} p_{\hat{\alpha}, \bar{L}}^{-1} (p_{\alpha, \bar{L}}^{-1})^{-1})^{-1} p_{\alpha, \bar{L}}^{-1}$. A similar argument shows that $p_{\alpha, \bar{L}}^{-1} (p_{\hat{\alpha}, \bar{L}} p_{\alpha, L})^{-1} p_{\hat{\alpha}, \bar{L}}^{-1}$ admits a holomorphic extension to $\mathbb{P}^1 \setminus \{\pm \alpha\}$. But

$$p_{\alpha, \bar{L}}^{-1} p_{\hat{\alpha}, \bar{L}} p_{\alpha, L}^{-1} p_{\hat{\alpha}, \bar{L}}^{-1} = (p_{\alpha, L}^{-1} p_{\hat{\alpha}, \bar{L}}^{-1} (p_{\alpha, \bar{L}}^{-1})^{-1})^{-1} p_{\alpha, \bar{L}}^{-1}.$$

We conclude that $p_{\alpha, \bar{L}}^{-1} p_{\hat{\alpha}, \bar{L}} p_{\alpha, L}^{-1} p_{\hat{\alpha}, \bar{L}}^{-1}$ extends holomorphically to \mathbb{P}^1 and is, therefore, constant. Evaluating at $\lambda = 0$ gives

$$p_{\alpha, \bar{L}}^{-1} p_{\hat{\alpha}, \bar{L}} p_{\alpha, L}^{-1} p_{\hat{\alpha}, \bar{L}}^{-1} = p_{\hat{\alpha}, \bar{L}}(0) p_{\alpha, \bar{L}}^{-1}(0),$$

completing the proof. \square

We proceed now to the proof of Theorem 4.2:

Proof. Consider projections $\pi_{\bar{L}} : \mathbb{C}^{n+2} \rightarrow \bar{L}$, $\pi_{(\bar{L} \oplus \rho \bar{L})^\perp} : \mathbb{C}^{n+2} \rightarrow (\bar{L} \oplus \rho \bar{L})^\perp$ and $\pi_{\rho \bar{L}} : \mathbb{C}^{n+2} \rightarrow \rho \bar{L}$ with respect to the decomposition

$$\mathbb{C}^{n+2} = \bar{L} \oplus (\bar{L} \oplus \rho \bar{L})^\perp \oplus \rho \bar{L}.$$

Since \bar{L} and $\rho \bar{L}$ are never orthogonal, condition (4.1) establishes, in particular, $\pi_{\rho \bar{L}} p(\alpha) = 0$. On the other hand, in view of (3.1b) and (3.1c), we have

$$(4.3) \quad \rho p(\lambda) = (-1)^{d+1} p(-\lambda)$$

for all λ . Hence

$$\pi_{\bar{L}} p(-\alpha) = (-1)^{d+1} \pi_{\bar{L}} \rho p(\alpha) = (-1)^{d+1} \rho \pi_{\rho \bar{L}} p(\alpha) = 0.$$

It follows that

$$p_{\hat{\alpha}, \bar{L}}(\bar{\lambda}^{-1}) p(\lambda) = \frac{\bar{\lambda}^{-1} - \bar{\alpha}^{-1}}{\bar{\lambda}^{-1} + \bar{\alpha}^{-1}} \pi_{\bar{L}} p(\lambda) + \pi_{(\bar{L} \oplus \rho \bar{L})^\perp} p(\lambda) + \frac{\bar{\lambda}^{-1} + \bar{\alpha}^{-1}}{\bar{\lambda}^{-1} - \bar{\alpha}^{-1}} \pi_{\rho \bar{L}} p(\lambda)$$

has no poles and, therefore, that

$$\hat{p}(\lambda) = r(1)^{-1} p_{\alpha, \bar{L}}^-(\bar{\lambda}^{-1}) p_{\hat{\alpha}, \bar{L}}(\bar{\lambda}^{-1}) p(\lambda)$$

has, at most, poles at $\lambda = \pm\alpha$.

Consider now projections $\pi_L : \mathbb{C}^{n+2} \rightarrow L$, $\pi_{(L \oplus \rho L)^\perp} : \mathbb{C}^{n+2} \rightarrow (L \oplus \rho L)^\perp$ and $\pi_{\rho L} : \mathbb{C}^{n+2} \rightarrow \rho L$ with respect to the decomposition

$$\mathbb{C}^{n+2} = L \oplus (L \oplus \rho L)^\perp \oplus \rho L.$$

By (3.1a), we have

$$(4.4) \quad \overline{p(\lambda)} = p(\bar{\lambda}^{-1}),$$

for all λ . In particular, $\overline{p(\alpha)} = p(\hat{\alpha})$ and, therefore, condition (4.1) establishes $p(\hat{\alpha}) \in \Gamma(L^\perp)$. Since L and ρL are never orthogonal, we conclude that $\pi_{\rho L} p(\hat{\alpha}) = 0$. Hence, by (4.3),

$$\pi_L p(-\hat{\alpha}) = (-1)^{d+1} \pi_L \rho p(\hat{\alpha}) = (-1)^{d+1} \rho \pi_{\rho L} p(\hat{\alpha}) = 0.$$

It follows that

$$p_{\alpha, L}^-(\bar{\lambda}^{-1}) p(\lambda) = \frac{\alpha - \bar{\lambda}^{-1}}{\alpha + \bar{\lambda}^{-1}} \pi_L p(\lambda) + \pi_{(L \oplus \rho L)^\perp} p(\lambda) + \frac{\alpha + \bar{\lambda}^{-1}}{\alpha - \bar{\lambda}^{-1}} \pi_{\rho L} p(\lambda)$$

has no poles and, then, by Lemma 4.3, that

$$\hat{p}(\lambda) = r(1)^{-1} K p_{\hat{\alpha}, \bar{L}}(\bar{\lambda}^{-1}) p_{\alpha, L}^-(\bar{\lambda}^{-1}) p(\lambda)$$

has, at most, poles at $\lambda = \pm\hat{\alpha}$. We conclude that $\hat{p}(\lambda)$ has no poles. Write $p(\lambda) = \sum_{k=-d}^d p_k \lambda^k$. The fact that

$$\lim_{\lambda \rightarrow \infty} \lambda^{-d} \hat{p}(\lambda) = r(1)^{-1} r(0) p_d$$

and

$$\lim_{\lambda \rightarrow 0} \lambda^d \hat{p}(\lambda) = r(1)^{-1} r(\infty) \overline{p_d}$$

are both finite establishes then $\hat{p}(\lambda)$ as a Laurent polynomial with degree smaller or equal to d .

Now let \hat{p} denote reflection across \hat{S} . According to (2.4), we have

$$(4.5) \quad r(-\lambda) = \rho r(\lambda) \rho^{-1},$$

for all $\lambda \in \mathbb{P}^1$, so that

$$\hat{p} \hat{p}(\lambda) = r(1)^{-1} \rho r(1) \hat{p}(\lambda) = r(1)^{-1} \rho r(\bar{\lambda}^{-1}) p(\lambda) = r(1)^{-1} r(-\bar{\lambda}^{-1}) \rho p(\lambda)$$

and, therefore, following (4.3),

$$\hat{p} \hat{p}(\lambda) = (-1)^{d+1} \hat{p}(-\lambda),$$

showing that the coefficients on λ^k in $\hat{p}(\lambda)$ are sections of \hat{S}^\perp if k has the same parity as d , being, otherwise, sections of \hat{S} .

Next we verify that $\overline{\hat{p}(\lambda)} = \hat{p}(\bar{\lambda}^{-1})$, equivalent to the complex conjugation conditions on the coefficients in $\hat{p}(\lambda)$. For that, observe that, by Lemma 4.3,

$$\begin{aligned} \overline{r(\lambda)} &= \overline{p_{\alpha, \bar{L}}^-(\lambda) p_{\hat{\alpha}, \bar{L}}(\lambda)} \\ &= p_{\alpha, \bar{L}}^-(\bar{\lambda}) p_{\alpha^{-1}, L}(\bar{\lambda}) \\ &= p_{\hat{\alpha}, \bar{L}}(\bar{\lambda}^{-1}) p_{\alpha, L}^-(\bar{\lambda}^{-1}) \\ &= K^{-1} r(\bar{\lambda}^{-1}), \end{aligned}$$

as well as, on the other hand,

$$\begin{aligned}\overline{r(\lambda)} &= \overline{K p_{\alpha, \bar{L}}(\lambda) p_{\alpha, L}^{-1}(\lambda)} \\ &= \overline{K p_{\alpha^{-1}, \bar{L}}(\bar{\lambda}) p_{\alpha, L}^{-1}(\bar{\lambda})} \\ &= \overline{K p_{\alpha, \bar{L}}^{-1}(\bar{\lambda}^{-1}) p_{\alpha, L}(\bar{\lambda}^{-1})} \\ &= \overline{K r(\bar{\lambda}^{-1})},\end{aligned}$$

for all $\lambda \in \mathbb{C} \setminus \{0, \pm\alpha\}$. In particular, $\overline{r(1)^{-1}} = r(1)^{-1} K$. The conclusion now follows immediately from (4.4).

Finally, note that

$$d_{\hat{S}}^{\lambda, \hat{q}} \hat{p}(\lambda) = r(1)^{-1} \circ r(\lambda) \circ d_S^{\lambda, q} \circ r(\lambda)^{-1} \circ r(\bar{\lambda}^{-1}) p(\lambda) = r(1)^{-1} r(\lambda) \circ d_S^{\lambda, q} p(\lambda) = 0,$$

for $\lambda \in S^1$, which completes the proof (since $d_{\hat{S}}^{\lambda, \hat{q}} \hat{p}(\lambda)$ is a polynomial with an infinite number of zeros). \square

Following Theorem 4.1 and Theorem 4.2, we have, furthermore:

Theorem 4.5. *Both constrained Willmore spectral deformation and Bäcklund transformation preserve the parallelism of the top term of a polynomial conserved quantity, for special choices of parameters.*

Proof. Suppose first that we are in the conditions of Theorem 4.1. Write $p(\lambda) = \sum_{k=-d}^d p_k \lambda^k$. Let π_{S^\perp} and $\pi_{\phi_q^\mu S^\perp}$ denote the orthogonal projections of $\mathbb{R}^{n+1,1}$ onto S^\perp and $(S_{\phi_q^\mu \Lambda})^\perp = \phi_q^\mu S^\perp$, respectively. Suppose that $p_d \in \Gamma(S^\perp)$ is parallel, $\pi_{S^\perp} \circ dp_d = 0$, and let us prove that then so is $\mu^d \phi_q^\mu p_d \in \Gamma((S_{\phi_q^\mu \Lambda})^\perp)$,

$$\pi_{\phi_q^\mu S^\perp} \circ d(\mu^d \phi_q^\mu p_d) = 0.$$

For that, note that, as $p_d \in \Gamma(S^\perp)$, we have $\mathcal{N} p_d \in \Gamma(S)$ and, on the other hand, $q p_d = 0$, since $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$. Thus

$$\pi_{S^\perp} \circ d_q^\mu p_d = \mathcal{D} p_d = \pi_{S^\perp} \circ dp_d = 0.$$

The fact that $\phi_q^\mu : (\mathbb{R}^{n+1,1}, d_q^\mu) \rightarrow (\mathbb{R}^{n+1,1}, d)$ preserves connections establishes then

$$\pi_{\phi_q^\mu S^\perp} \circ d \circ \phi_q^\mu p_d = \phi_q^\mu \circ \pi_{S^\perp} \circ d_q^\mu p_d = 0.$$

The conclusion follows, by the constancy of μ .

Suppose now that we are in the conditions of Theorem 4.2 for $p(\lambda) = \sum_{k=-d}^d p_k \lambda^k$. Suppose, again, that $p_d \in \Gamma(S^\perp)$ is parallel, $\mathcal{D} p_d = 0$, and let us prove that then so is

$$\hat{p}_d := \lim_{\lambda \rightarrow \infty} \lambda^{-d} \hat{p}(\lambda) = r(1)^{-1} r(0) p_d,$$

the top term of $\hat{p}(\lambda)$, $\mathcal{D}_{\hat{S}} \hat{p}_d = 0$. In view of Proposition 3.4, we are left to verify that $\mathcal{D}_{\hat{S}}^{1,0} \hat{p}_d = 0$ or, equivalently, that $r(1)^{-1} \circ (\mathcal{D}_{\hat{S}}^{1,0})^{1,0} \circ r(0) p_d = 0$, for

$$\hat{d} := r(1) \circ d \circ r(1)^{-1}.$$

But

$$(\mathcal{D}_{\hat{S}}^{1,0})^{1,0} = r(0) \circ (\mathcal{D}^{1,0} - q^{1,0}) \circ r(0)^{-1} - \tilde{q}^{1,0}$$

(see [10], Lemma 3.9). Now note that, evaluating (4.5) at $\lambda = 0$ and at $\lambda = \infty$ shows that both $r(0)$ and $r(\infty)$ commute with ρ , establishing, in particular, that

$$r(0)|_{S^\perp}, r(\infty)|_{S^\perp} \in \Gamma(O(S^\perp)).$$

The fact that $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ vanishes in $\Gamma(S^\perp)$ together with the parallelism of p_d combine to complete the proof. \square

For the particular case of $d = 1$, it follows that:

Corollary 4.6. *The class of parallel mean curvature vector surfaces in space-forms is preserved under both constrained Willmore spectral deformation and Bäcklund transformation, for special choices of parameters, with preservation of the space-form in the latter case.*

The class of constant mean curvature surfaces in 3-dimensional space-forms is preserved under both constrained Willmore spectral deformation and Bäcklund transformation, for special choices of parameters, with preservation of both the space-form and the mean curvature, in the latter case.

Proof. The preservation of the space-form under Bäcklund transformation is a consequence of the fact that, in the conditions of Theorem 4.2, $\hat{p}(1) = p(1)$. Assuming now that $n = 3$, we can take unit sections N and \hat{N} such that $S^\perp = \langle N \rangle$ and $\hat{S}^\perp = \langle \hat{N} \rangle$. Considering the sections $\beta \in \Gamma(\mathbb{C})$ and $\hat{\beta} \in \Gamma(\mathbb{C})$ such that $p_1 = \beta N$ and $\hat{p}_1 = \hat{\beta} \hat{N}$, we get, by virtue of $(\hat{p}(\lambda), \hat{p}(\lambda)) = (p(\lambda), p(\lambda))$ (recall Theorem 4.2), that $\hat{\beta}^2 = (\hat{p}_1, \hat{p}_1) = (p_1, p_1) = \beta^2$ (which is constant). Therefore $\hat{\beta} = \beta$ or $\hat{\beta} = -\beta$. According to Theorem 3.14, we obtain the preservation of the mean curvatures of Λ and $\hat{\Lambda}$ in $S_{p(1)} = S_{\hat{p}(1)}$, since

$$(\operatorname{Re}(\hat{p}_1), \operatorname{Re}(\hat{p}_1)) = (\operatorname{Re}(\hat{\beta}))^2 = (\operatorname{Re}(\beta))^2 = (\operatorname{Re}(p_1), \operatorname{Re}(p_1)).$$

□

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